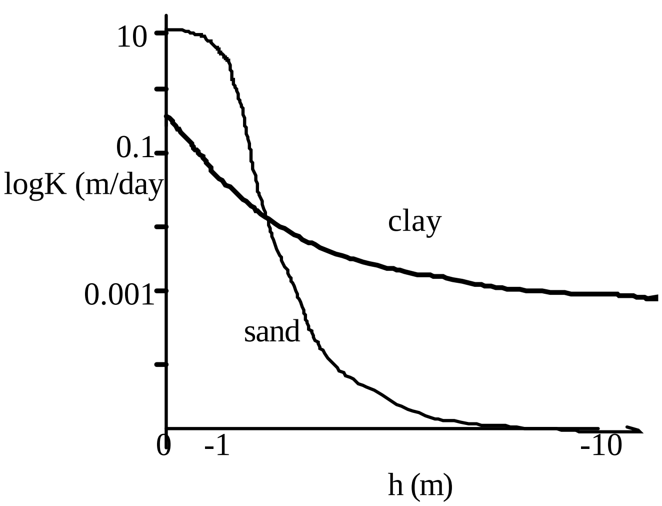
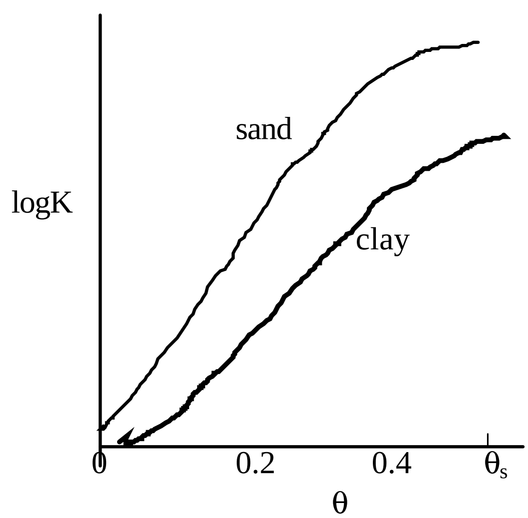
Chapter 4 - Water Flow in Unsaturated Soils

- Unsaturated hydraulic conductivity
- Steady water flow in unsaturated soil
- Soil-water pressure and total head distributions
- Units of K
- Flux and velocity
- Transient water flow

Unsaturated Soil hydraulic conductivity

As some of soil pores empty, the ability of soil to conduct H_20 decreases drastically

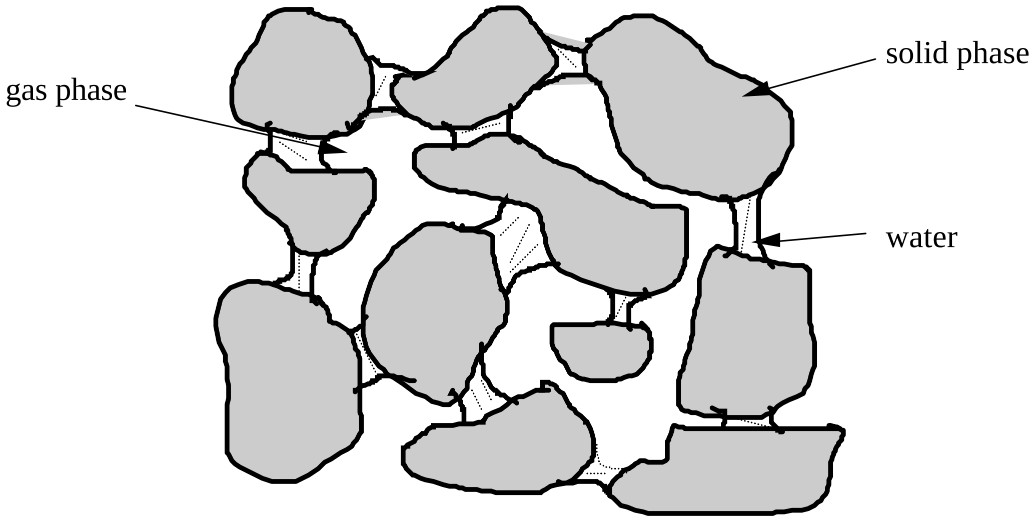




Thus, K is a function of h or θ

Or: $K = f(h)$ and $K = f(\theta)$ ---→ highly nonlinear function

As some of the soil pores empty, the ability of soil to conduct water decreases drastically:



Water in unsaturated soil.

Unsaturated hydraulic conductivity decreases as volumetric water content decreases:

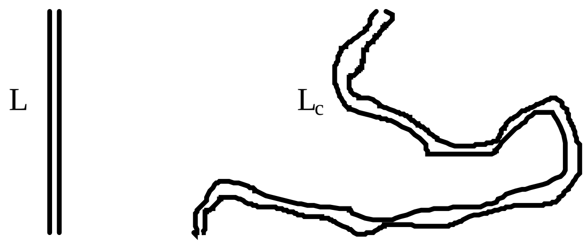
- Cross-sectional area of water flow decreases;
- Tortuosity increases;
- Drag forces increase

Thus, the unsaturated hydraulic conductivity is a nonlinear function of θ and h.

✪ Explain why the saturated hydraulic conductivity for a coarse-textured soil is larger than a fine-textured soil.

Use capillary tube model (Poiseuille's law) to derive K(θ)

Assume soil pores consist of bundles of capillary tubes of different sizes (L_c = length of twisted capillary, with a length $L_c > L$ (length of column)



$$Q_J = \frac{p\ R^4\,\Delta P}{8Ln} = \frac{p\ R_J^4\ r_w\,g}{8n}\frac{\Delta H}{L_c} \quad \textit{for a single capillary tube J}$$

Then for the total volumetric flow rate (Q_T)

$$\boldsymbol{Q_T = \sum_{J=1}^{M} N_J\,Q_J = \frac{p\ r_w\,g}{8n}\frac{\Delta H}{L_c}\sum_{J=1}^{M} N_J\,R_J^4} \;\; \textbf{\textit{, where }} \boldsymbol{N_J} \textbf{\textit{ is number of capillaries of radius }} \boldsymbol{R_J}$$

and M is the number of different capillary size classes in the bundle of capillary tubes.

The total flux (J_w) can be calculated by dividing through by the total cross-sectional area of the soil column:

$$J_w = \frac{Q_T}{A} = \frac{p\ r_w\,g}{8n}\frac{\Delta H}{L_c}\sum_{J=1}^{M} n_J\,R_J^4 \;\; \textit{where } n_J = N_J / A \textit{ is number of}$$

$$\textit{capillaries per unit area of radius } R_J$$

While defining tortuosity as $\tau = L_c/L$, the flux equation can be rewritten as:

$$J_w = \frac{p\ r_w g}{8nt} \sum_{J=1}^{M} n_J R_J^4 \frac{DH}{L} = -K(q)\frac{DH}{DX}$$

(with $-\Delta X = L$), and where the unsaturated hydraulic conductivity value can be inferred from the soil water retention curve, with $n_J \pi R_J^2$ denoting the volume of water-filled pores that drain per unit volume of soil at a certain soil-water pressure head (h_J).

✪ What determines water flow in soil?

1) Total head gradient
2) K versus θ relationship

Hence, although the soil is unsaturated, we can still apply the Darcy equation

Hydraulic conductivity cm/hr	Water content %	soil-water pressure head cm
	Sand	
30	25	-0
0.15	20	-50
0.004	10	-100
	Loam	
2.8	45	0
1.3	44	-25
1.2	42	-50
0.17	35	-75
0.071	33	- 90
0.037	30	-100
0.0048	20	-1,000
0.00054	10	-14,000
0.00006	8	-15,000

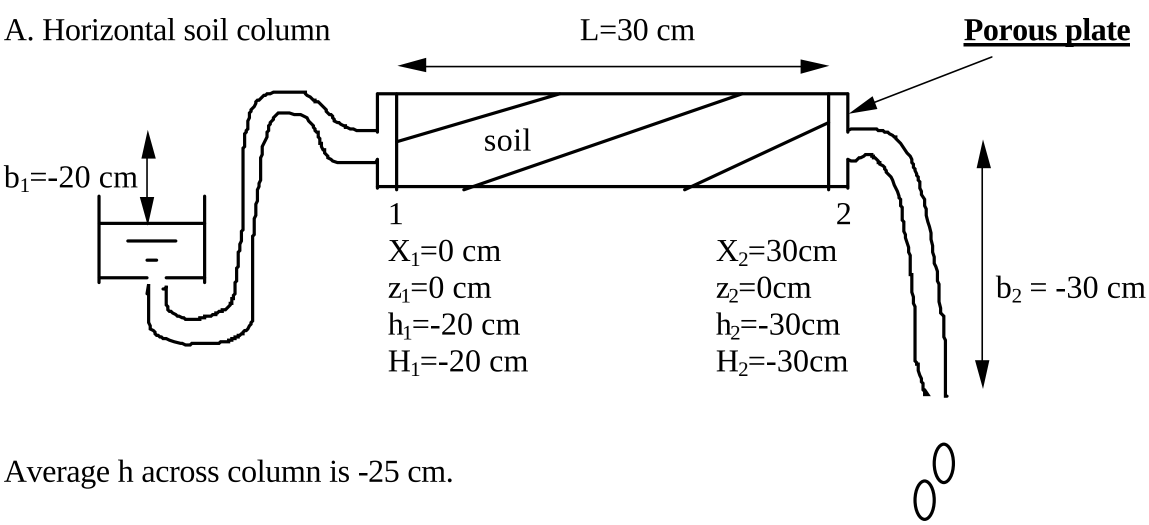
<u>What can we learn from these data?</u>

- Hydraulic conductivity decreases as the water content and/or soil water pressure head decreases;
- In general, for any volumetric water content, the unsaturated hydraulic conductivity for a coarser-textured soil is larger than for a finer-textured soil;
- In general, for any specific soil water pressure head (not close to saturation), the unsaturated hydraulic conductivity for a coarse-textured soil is smaller than for a finer-textured soil.

✪ What other soil hydraulic property is needed to explain this last point ?

<u>Examples of unsaturated steady flow experiments:</u>

A. Horizontal soil column



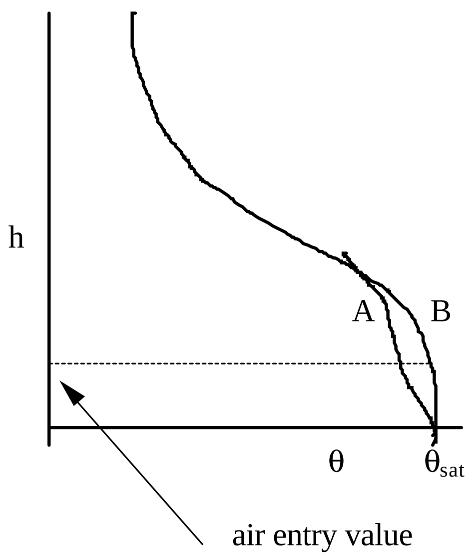
Average h across column is -25 cm.

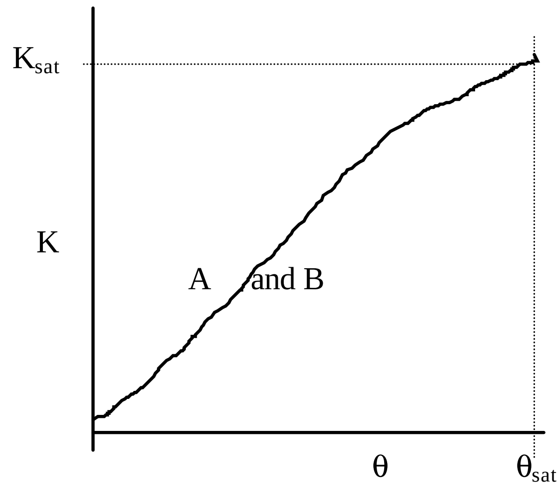
Note that soil is unsaturated (h <0), so we must use an average h to estimate K. This is acceptable, if changes in h are small. K at -25 cm is 1.3 cm/hr (see table for loamy soil). Then

$$J_w = -(1.3\ cm/hr)\,\frac{-30cm-(-20cm)}{30cm-0cm} = 0.43\ cm/hr$$

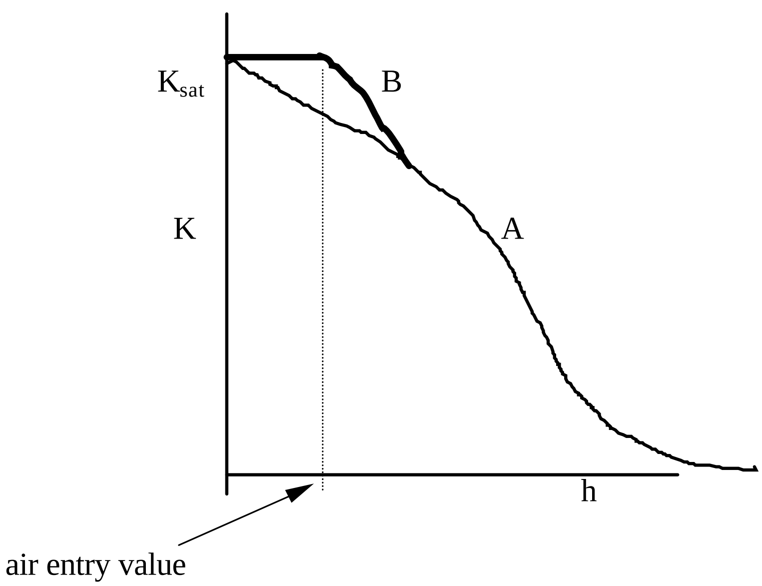
Definition of air-entry value

✪ For soils A and B, the soil water retention curves and $K(\theta)$-curves are given below. What is the difference between the two soils. Draw also the approximate $K(h)$-curve for soil B.



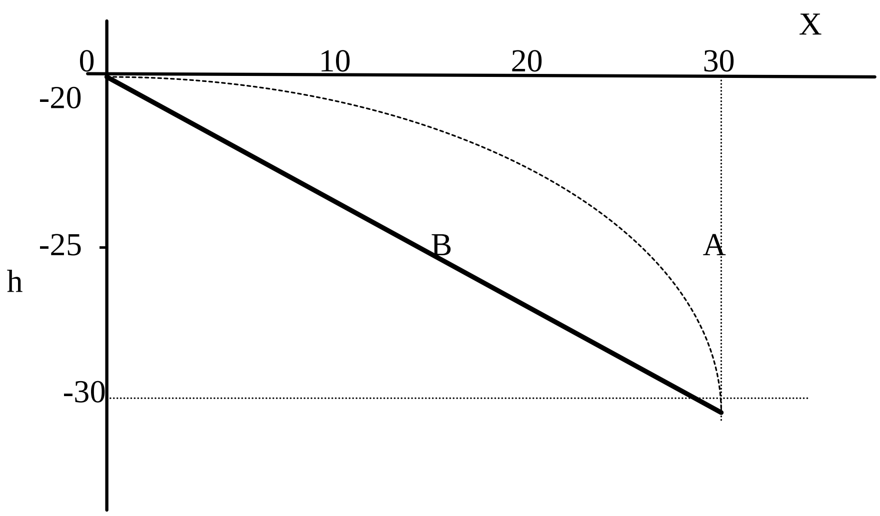


The air entry value of a specific soil is determined by the radius of the largest pore. If this largest pore is relatively small, the air entry value will be relatively large.



✪ Draw the approximate decrease in h along the horizontal soil column of page 4-5.

Do this for both soils A and B, if the air entry value is -30 cm

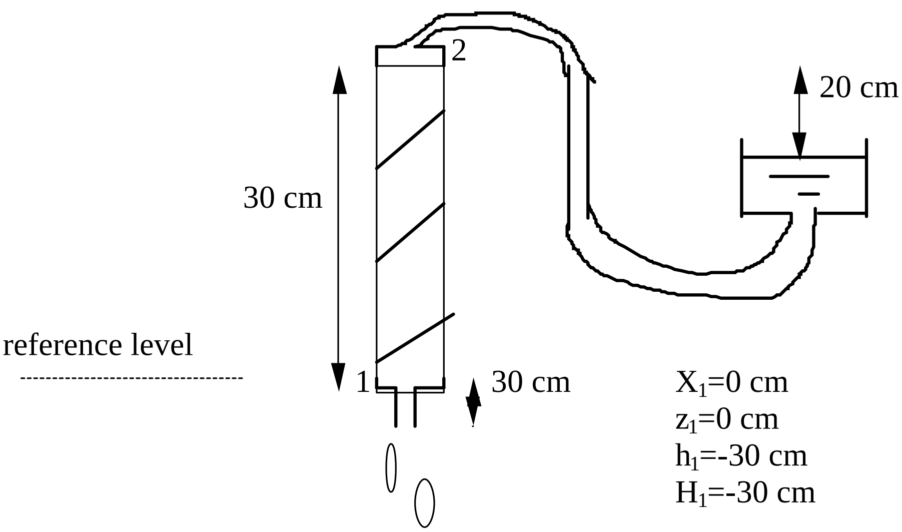


Sometimes one uses bubbling pressure to denote air entry value.

- ✪ What is the bubbling pressure of a one-bar ceramic porous plate ?
- ✪ What is the approximate bubbling pressure of the porous cup of a tensiometer ?

Remember that simple averaging of K will not work in a column with large DH, because K is not constant with X, and consequently D H/DX is nonlinear with X (we will get back to this later !).

B. Vertical soil column, with identical soil water pressure head values at boundary conditions:



X_1=0 cm	X_2=30cm
z_1=0 cm	z_2=30cm
h_1=-30 cm	h_2=-20cm
H_1=-30 cm	H_2= 10cm

With same average K-value as above:

$$J_w = -(1.3\ cm\ /\ hr)\ \frac{10cm - (-30cm)}{30cm - 0cm} = -1.73\ cm\ /\ hr$$

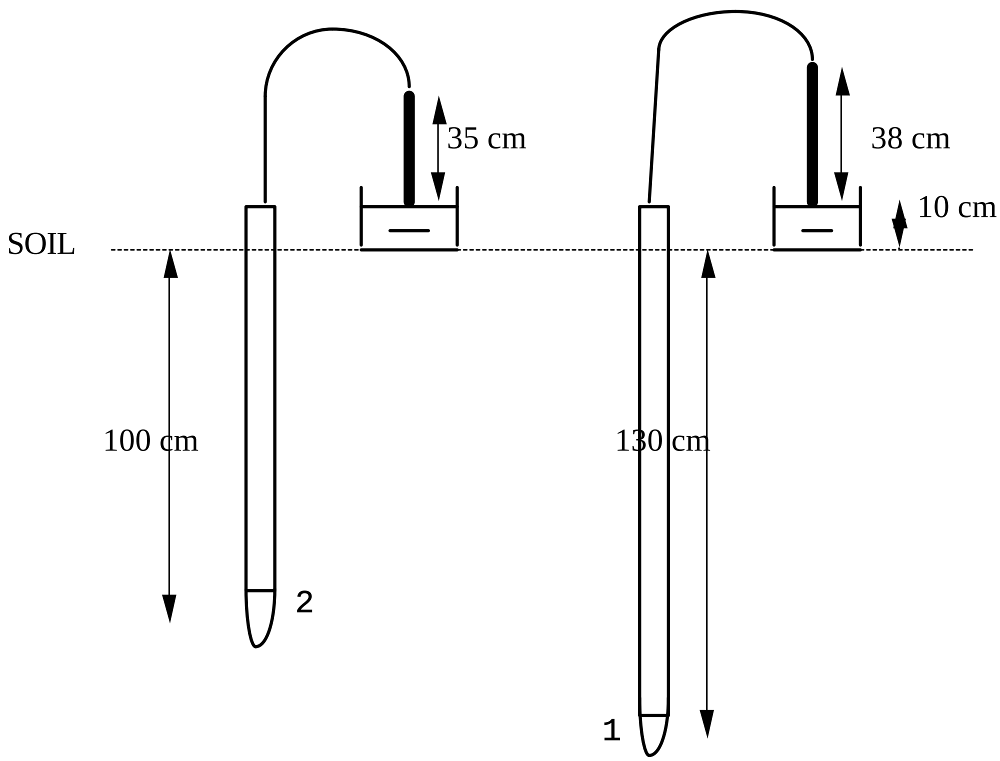
✪ Why is the flux density for the vertical column much larger than for the horizontal column ?

Now, change this problem so the water level in the reservoir is at the reference level (h_2 = -30 cm). Then, water flow is only due to gravity gradient (soil water pressure head gradient is 0), and with K (at -30 cm) approximately equal to 1.25 cm/hr, the flux is

$$J_w = -1.25 \frac{0-(-30)}{30-0} = -1.25\ cm\ /\ hr$$

NOTE: In this case, the soil-water pressure head is constant throughout the column, so the water content value is uniform as well. This is the best system to set up for determination of unsaturated K in the laboratory, because K is constant with depth.

Another Example: Two tensiometers in the field (close together):



X_2 = -100 cm
z_2 = -100 cm
h_2 = y-13.5X = 145-13.5(35)
= -327 cm
H_2 = -100 + (-327) = -427 cm

X_1 = -130 cm
z_1 = -130 cm
h_1 = 178-13.5(38) = -335 cm

H_1 = -130 + (-335) = -465 cm

Assume that the soil is uniform in the horizontal direction.

NOTES: $H_2 > H_1$, hence water is flowing from H_2 to H_1 (down). Under field conditions the soil surface is chosen for the gravity reference.

Soil-water pressure head is negative, so the soil is unsaturated and K is a function of h

Determine the average soil water pressure head:

$$\bar{h} = \frac{-327cm \ -335cm}{2} = -331cm \qquad \text{“field capacity”}$$

From a hypothetical table we find K_{-331} = 0.01 cm/hr; thus K(h) is known. Then:

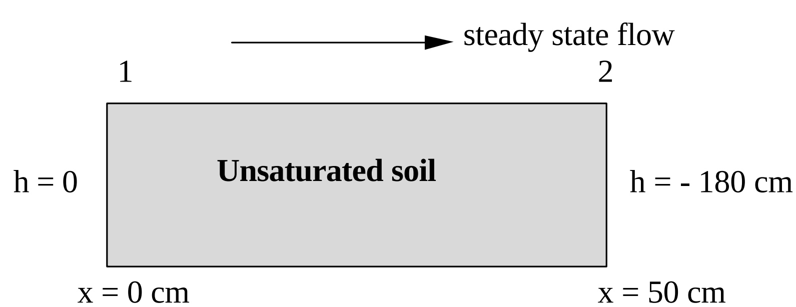
$$\boldsymbol{J_w = -\ (0.01\ cm\ /\ hr)\ \frac{-427 - (-465)}{-100\ \ -(-130)} = -\ .0127\ cm\ /\ hr} \quad \underline{\textbf{DOWN}}$$

$$\boldsymbol{= -\ .3\ cm\ /\ day}$$

The downward flux of .3 cm/day is significant, if compared with plant transpiration.

✪ What is the range in plant transpiration (mm/day) in California ?

<u>Soil-water pressure head distributions for steady state flow in unsaturated soils:</u>



Problem: How does the soil-water pressure head vary with x?
NOTE: Both h and K(h) changes greatly with x inside column.

Without calculus: $J_w = -K \dfrac{H_2 - H_1}{x_2 - x_1}$

With calculus $J_w = -K \dfrac{dH}{dx}$

$$H = h + z$$

$$\frac{dH}{dx} = \frac{dh}{dx} + \frac{dz}{dx}$$

$$J_w = -K \frac{dh}{dx}$$

$$\int_0^x J_w \, dx = -\int_0^h K \, dh$$

Assume $K = a + bh$ (cm/day)

Then:

$$\int_0^x J_w \, dx = -\int_0^h (a + bh) dh$$

$$J_w x \Big|_0^x = -\left[ah + \frac{bh^2}{2} \right]_0^h = -ah - \frac{bh^2}{2}$$

$$J_w x = -ah - \frac{b}{2} h^2$$

$$\frac{b}{2} h^2 + ah + J_w x = 0$$

This is a quadratic equation, which can be easily solved for h:

✪ How is the following solution obtained ?

$$h = -\frac{1}{b} \left(a \pm \sqrt{a^2 - 2bJ_w x} \right)$$

The solution has both a positive and negative root. Which one can only be used?

At x=0, h = 0, so we can only use the negative root:

$$h = -\frac{1}{b}\left(a - \sqrt{a^2 - 2bJ_w x}\right)$$

This solution yields h as a function of distance x, or h = f(x).
However, we must first know J_w
This can be determined, if K as a function of h is known.

Let a = 4, b = 0.02. Then use previously found relationship:

$$\int_0^x J_w\, dx = -\int_0^h (a + bh)dh$$

And substitute the known values of h at the left and right-hand side of the soil column:

$$J_w x \Big|_0^{50} = -\left[a h + \frac{bh^2}{2}\right]_0^{-180}$$

Substitute the known values of a and b, and solve for J_w:

$$50\ J_w = 180\left[4 + \frac{0.02}{2}(-180)\right]$$

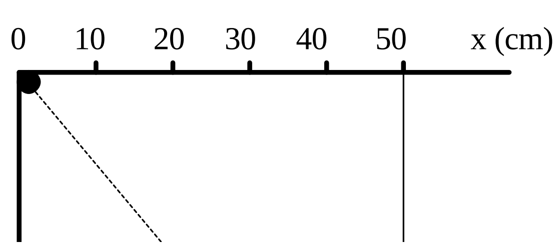
J_w = 7.92 cm/day

Now go back and substitute the values of a, b, and J_w in above:

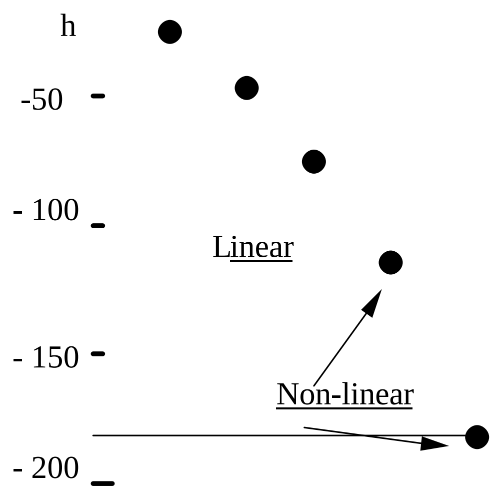
$$h = -\frac{1}{0.02}\left(4 - \sqrt{16 - 2(.02)(7.92)x}\right)$$

The following graph shows the change of h from the left-hand side (x=0) to the right-hand side of the soil column:

x (cm) h (cm)



0	0
10	-20.9
20	-44.6
30	-72.6
40	-108.8
50	-180.0

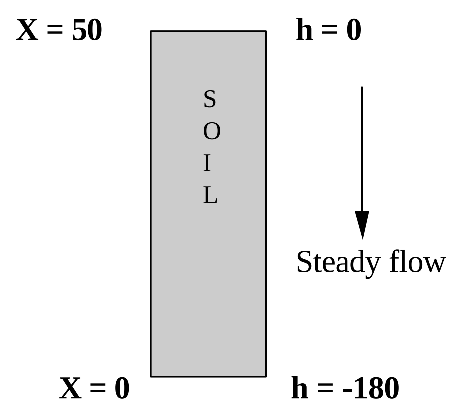


✪ Explain the nonlinear change of h with x ?

<u>A similar derivation can be written for the **vertical case** (with identical boundary conditions):</u>

Assume again that K = a + bh (with a=4 and b=0.02), and solve for J_w and h(z):

$$J_w = -K\frac{dH}{dX} \qquad (X{=}z)$$



$$H = h + z$$

$$\frac{dH}{dz} = \frac{dh}{dz} + \frac{dz}{dz}$$

$$J_w = -K\frac{dh}{dz} - K$$

$$\frac{J_w}{K} + 1 = -\frac{dh}{dz}$$

$$dz = -\frac{dh}{(1 + J_w/K)} = -\left(\frac{K}{K+J_w}\right)dh$$

Still not in easy form for integration. Change to:

$$dz = -\left(\frac{K+J_w-J_w}{K+J_w}\right)dh = -\left(1 - \frac{J_w}{K+J_w}\right)dh$$

Substitute for K and integrate:

$$\int_0^{50} dz = -\int_{-180}^{0} dh + \frac{J_w}{b}\int_{-180}^{0} \frac{bdh}{a + bh + J_w}$$

$$\int \frac{du}{u} = \ln u + c, \; let\ u = a + bh + J_w$$

$$which\ gives\ du = bdh$$

$$50 = -180 + \frac{J_w}{b}\ln(a + bh + J_w)\Big|_{-180}^{0}$$

$$230 = \frac{J_w}{b}\ln\left(\frac{a + J_w}{a - 180b + J_w}\right)$$

$$\frac{230b}{J_w} = \ln\left(\frac{a + J_w}{a - 180b + J_w}\right)$$

$$a = 4,\ b = 0.02$$

$$\frac{4.6}{J_w} = \ln\left(\frac{4 + J_w}{0.4 + J_w}\right)$$

Get J_w by iteration or graphically to obtain: J_w = -10.71 cm/day.

Then, to find h(z):

$$\int_0^z dz = -\int_{-180}^{h} dh + \frac{J_w}{b}\int_{-180}^{h} \frac{b\,dh}{a + bh + J_w} \qquad z = -h - 180 + \frac{J_w}{b}\ \ln\left(\frac{a + bh + J_w}{a - 180b + J_w}\right)$$

which yields with a=4, b=0.02 and J_w= -10.71: z = -180 - h - 535.5 ln(0.651 - 0.00194h)

We can't solve for h directly, but h(z) can be obtained by substitution of h-values between 0 and -180, to compute the appropriate z-values.

Head Profiles

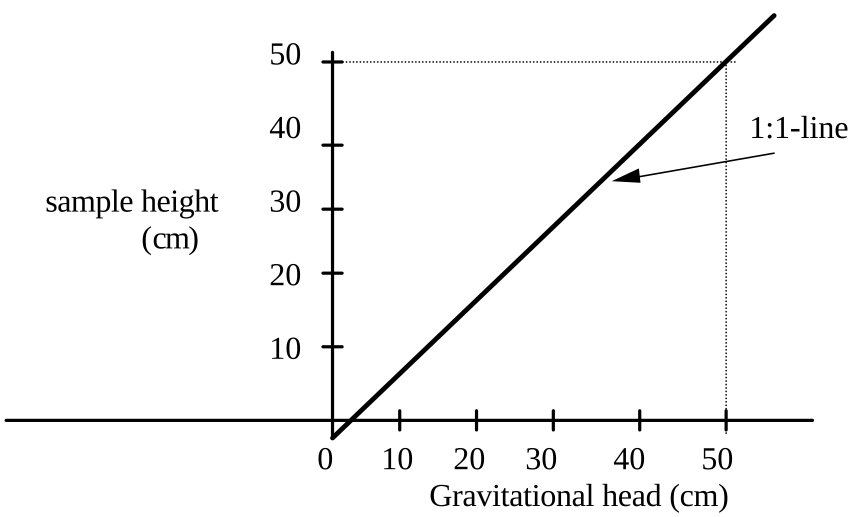
- Describe the distribution of gravitational head (z), soil-water pressure head (h), and total head (H);

- Used to determine direction of water flow in a soil system

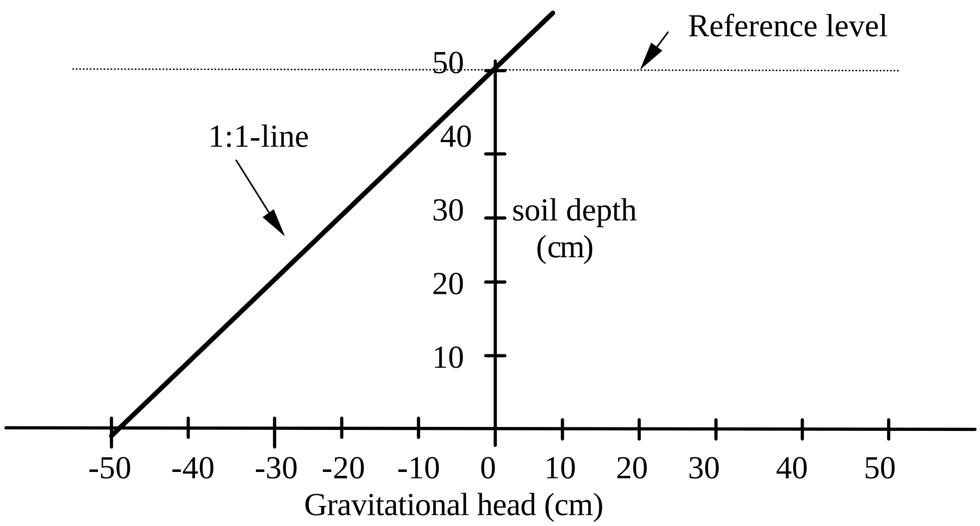
H = h + z , where z is determined by the height of the point of interest relative to some reference plane,

Gravitational head

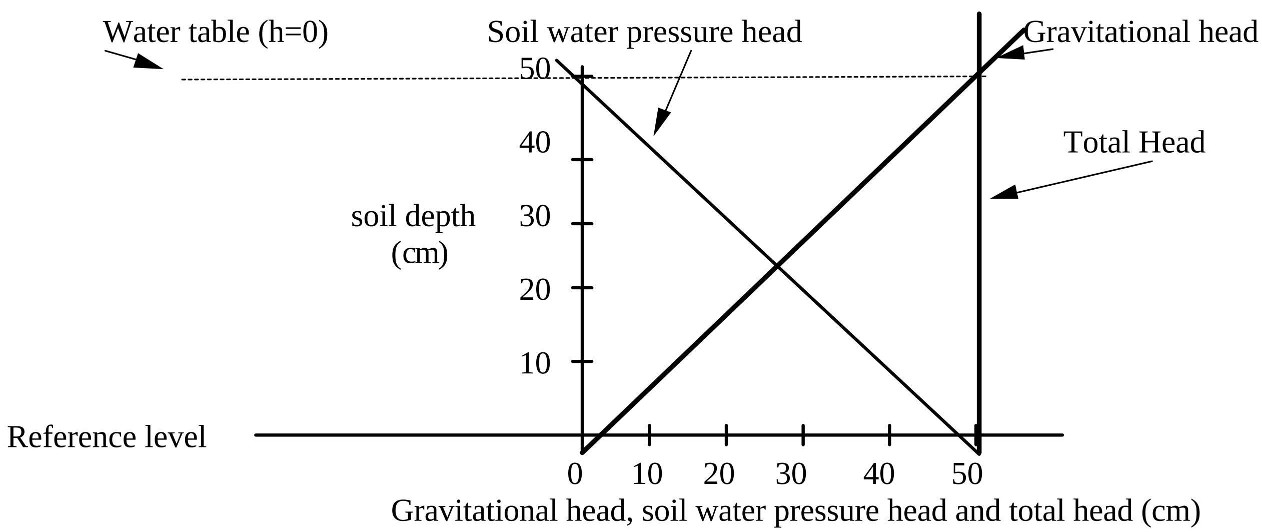
If the **reference level is taken at the bottom of a soil column** (or below the soil surface), we plot the gravitational head as follows:



If the **reference level is taken at the top of the sample** (or at the soil surface), the slope of the line is the same, but the values of the gravitational head are negative as follows:

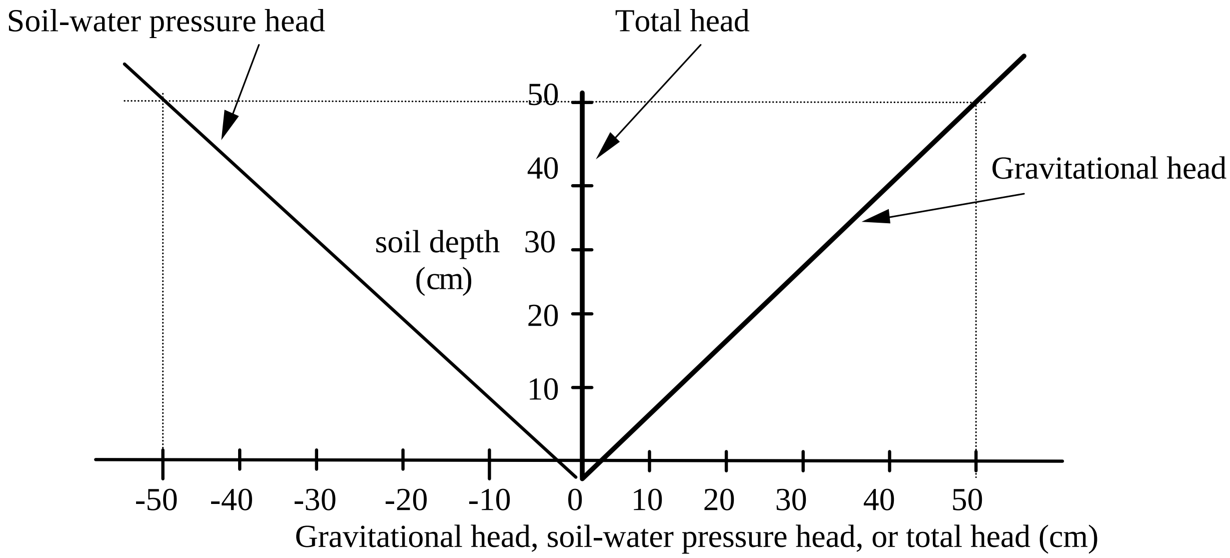


Below the water table; No water Flow: H_1 (0 cm) = H_2 (50 cm)



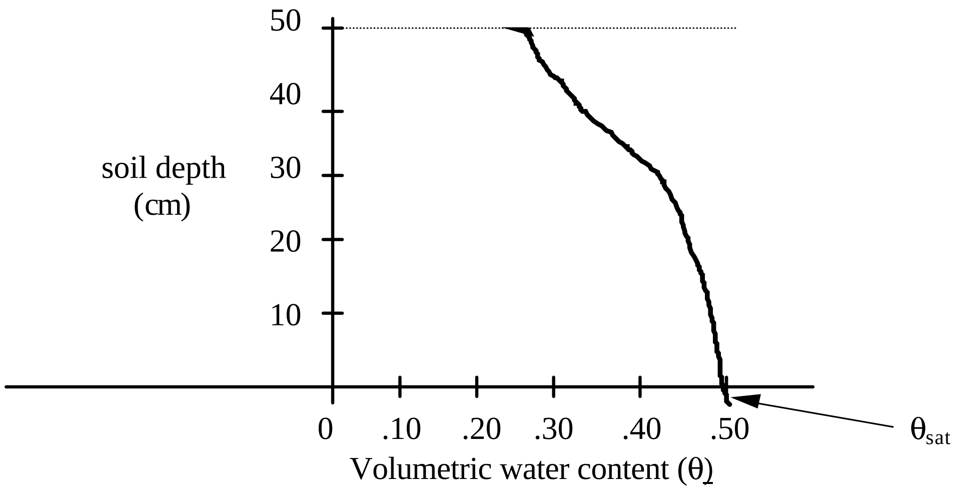
Note: in this example in the field or in a column, the reference level is selected at 50 cm below the water table. You can place a gravity reference anywhere as long as it was not already specified in a problem and provided you keep the same location throughout the working of the problem.

<u>No water flow (hydraulic equilibrium) above water table; reference level at water table.</u>



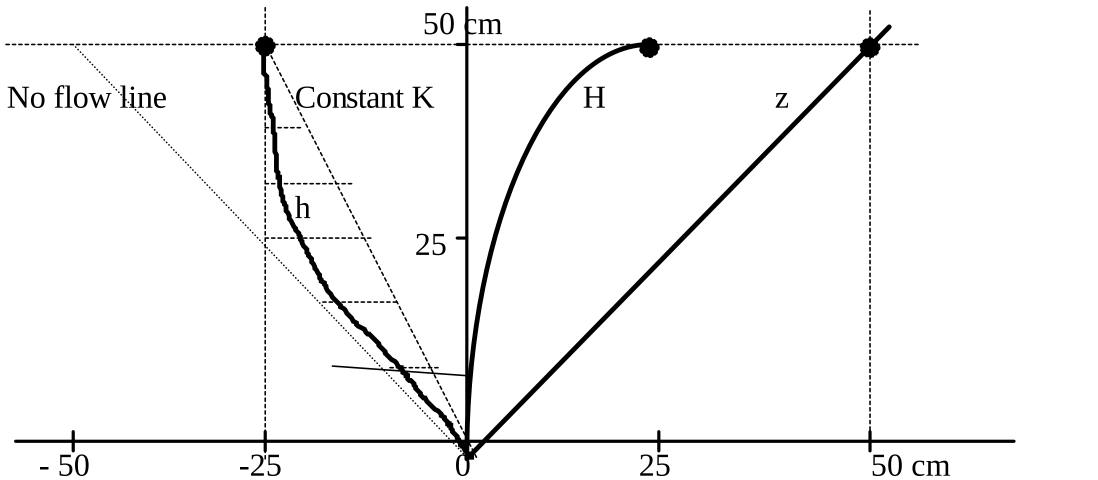
✪ What is the soil water pressure head at the water table ?

<u>q for previous case</u>



✪ Which soil hydraulic property is needed to determine the change of volumetric water content with soil depth from the hydraulic equilibrium profile ?

Steady rainfall, steady-state vertical flow (J_w constant)



Given that:

- Water flow is steady state in the downward direction;
- Water table at 0 cm (h=0);
- Reference level is selected: elevation is 0 cm;
- The soil-water potential at the soil surface is measured and known (see symbol);

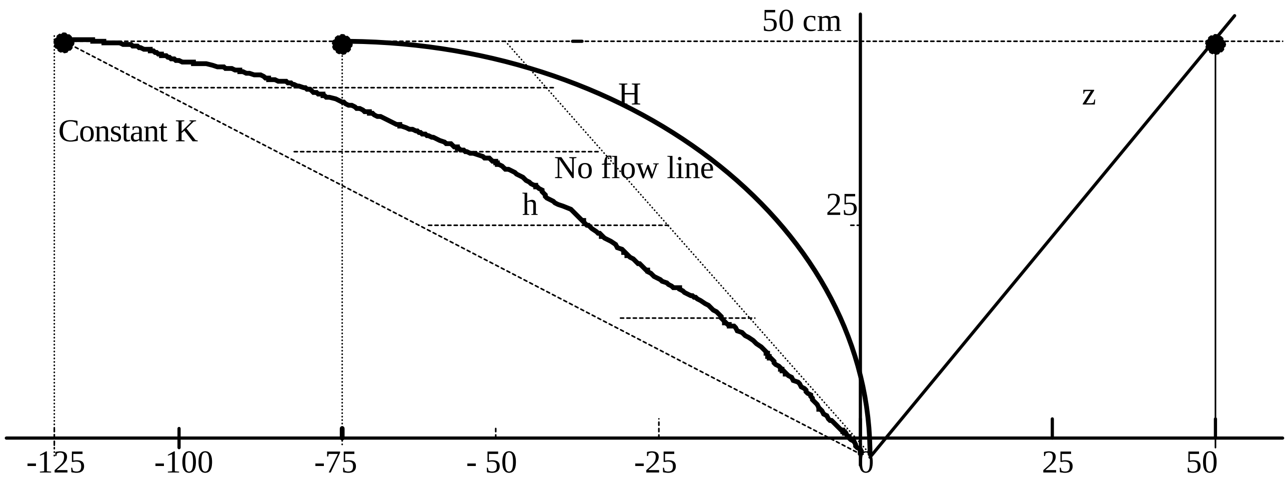
  ✪ Plot the distribution of h and H with position ?

The "no flow" line is the <u>hypothetical</u> h- line if indeed water flow is not occurring. The "Constant K" line is the <u>hypothetical</u> h-line if K is indeed a constant throughout the column. However, K cannot be a constant for an unsaturated column except for the case where h is maintained everywhere the same within the column or h > air entry value. The vertical broken line at h = -25 cm further defines the region in which the real h must lie. The real h must fall within the area bounded by the "Constant K", the "no flow", and the vertical broken line.

At top of soil: h is the smallest (most negative), hence the K is the smallest. Therefore, the hydraulic head gradient ($\Delta H/\Delta X$) is largest there.

At the bottom of the soil: h is the largest (least negative), hence the K is the largest. Therefore, the hydraulic head gradient ($\Delta H/\Delta X$) is smallest there.

Steady state evaporation (J_w constant)



As for the steady state rainfall situation:

At top of soil: h is the smallest (most negative), hence the K is the smallest. Therefore, the hydraulic head gradient ($|\Delta H/\Delta X|$) is largest there.

At the bottom of the soil: h is the largest (least negative), hence the K is the largest. Therefore, the hydraulic head gradient ($|\Delta H/\Delta X|$) is smallest there.

<u>Determining shape of h and H versus. X for steady flux situations</u>

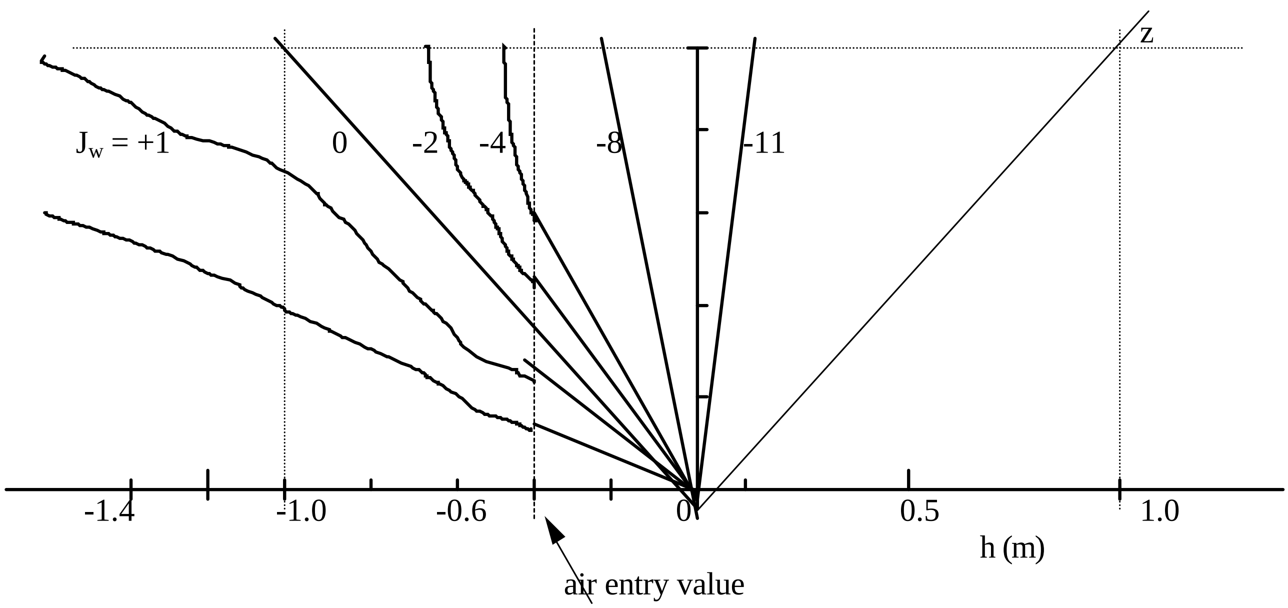
1. Plot z, h and H at top and bottom of profile (H=h+ z)

2. If $h \geq 0$ (i.e. saturated), draw straight lines. (Lines are also straight inside the capillary fringe).
3. For $h < 0$, determine whether K is smaller or larger at one end as compared to the other. K is largest where h is greatest.

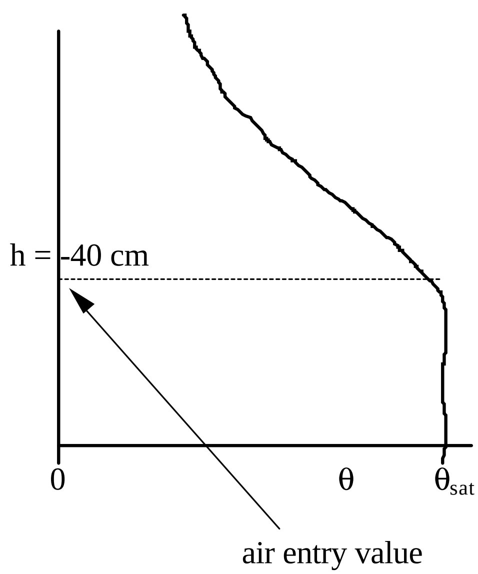
4. Since J_w must be everywhere constant, determine whether $|\Delta H/\Delta X|$ is larger or smaller at one end as compared to the other end. If K small, then $|\Delta H/\Delta X|$ must be large. If K large, then $|\Delta H/\Delta X|$ must be small.

5. By knowing the relative magnitudes of $\Delta H/\Delta X$ throughout the profile, plot H versus X by changing slope accordingly. (The curve drawn is only an approximation.)
6. With H versus X, you can get h versus. X. **TOTAL HEAD IS DRAWN FIRST!**

✪ The soil for which the diagram below is given has a saturated hydraulic conductivity of 10 cm/day, and has a constant water table at 1 m below the soil surface. Explain what you see in the diagram, which presents soil-water pressure head distributions for a range of flux density values.



And has the following soil water retention curve:



<u>What are units of K ?</u>

Using potential per unit mass

$$J_w = -K\frac{\Delta m_T}{\Delta X}$$

$$\frac{m}{\sec} = -\,?\,\frac{J}{kg\ m}$$

$$K = \frac{m\ \sec^{-1}}{J\ kg^{-1}\ m^{-1}} = \frac{kg\ m^2}{J\ \sec}$$

$$= \frac{kg\ m^2}{Nm\ \sec} = \frac{kg\ m^2\ \sec^2}{kg\ m^2\ \sec}$$

$$\therefore K = \underline{\sec}$$

Using potential per unit weight (head)

$$J_w = -K\frac{?H}{?X} \quad \text{or} \quad \frac{cm}{sec} = ?\frac{cm}{cm}$$

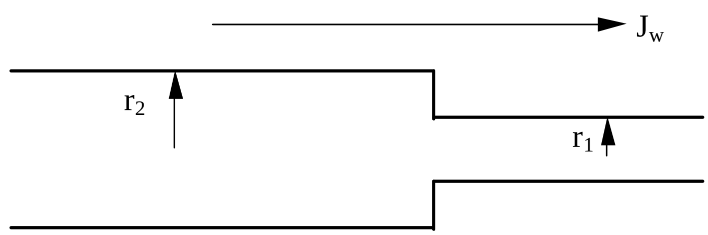
thus, units of **K** = **cm/sec**

✪ What are the units of K, if using potential per unit volume ?

Flux and velocity

$$J_w = \frac{V}{At} = \frac{cm^3 \; H_2O}{cm^2 \; soil\ surface \; \sec}$$

✪ What happens to velocity of H_2O when flowing from a large to a smaller diameter tube?

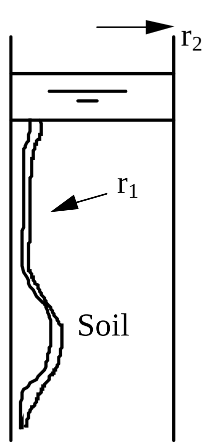


It increases by a factor of $\left[\frac{r_2}{r_1}\right]^2$ where $r_1 < r_2$

That is,

$Q = v_2A_2 = v_1A_1$, and $A=\pi r^2$ for a cylindrical tube. Then $v_1 = v_2A_2/A_1 = v_2(r_1/r_2)^2$

The same occurs in soils when water flows from large to small pores:



Cross-sectional area through which water moves decreases as it moves into the soil. Therefore, the water in the soil moves faster than on top of soil.

$\bar{v} = \frac{J_w}{q}$ = average pore water velocity

<u>Intrinsic permeability</u>, k

Removes fluid properties, such as density and viscosity, as factors influencing water flow, and yields a constant (property of the soil only).

This soil characteristic is defined as the soil permeability: $k = \dfrac{K}{f} = \dfrac{cm\ /\ \mathrm{sec}}{1\ /\ cm\ \mathrm{sec}} = \mathrm{cm}^2$

<u>Fluidity is fluid property</u>, f - ability of liquid or gas to flow, based on its viscosity and density;

$$f = \frac{rg}{n} \quad , \text{ where } \nu \text{ is dynamic viscosity, } \frac{N\ \mathrm{sec}}{m^2}$$

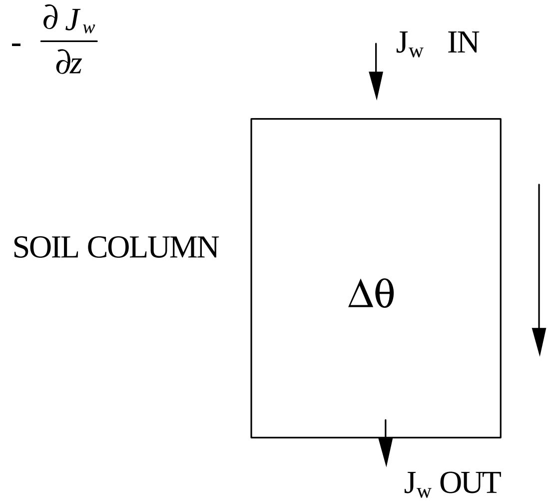
<u>Transient Vertical Water Flow</u>

$$J_w = -K\frac{dH}{dz} = -K\frac{d(h + z)}{dz}$$

$$J_w = -K\frac{dh}{dz} - K \quad \textit{at steady - state}$$

Transient state must include time - Equation of Continuity

$$\frac{\partial q}{\partial t} = -\frac{\partial J_w}{\partial z}$$



$$\frac{\partial q}{\partial t} = - \frac{\partial}{\partial z} (- K \frac{\partial h}{\partial z} - K)$$

$$\frac{\partial q}{\partial t} = - \frac{\partial}{\partial z} (- K \frac{\partial h}{\partial z}) + \frac{\partial K}{\partial z}$$

Convert h to θ

$$\frac{\partial h}{\partial z} = \frac{\partial h}{\partial q} \frac{\partial q}{\partial z}$$

$$\frac{\partial q}{\partial t} = - \frac{\partial}{\partial z}\left(- K \frac{\partial h}{\partial q} \frac{\partial q}{\partial z} \right) + \frac{\partial K}{\partial z}$$

$$Let\ D(q) = K \frac{\partial h}{\partial q}$$

$\frac{\partial h}{\partial q}$ is slope of soil-water characteristic curve, and D is soil water diffusivity (m^2/sec)

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial z} \left(D \frac{\partial q}{\partial z} \right) + \frac{\partial K}{\partial z} \qquad \textbf{Richards Equation}$$

Solutions

- Partial differential equations
- Boundary value problems
- Numerical Techniques
    - Finite difference
    - Finite element