Chapter 4 - Water Flow in Unsaturated Soils

- Unsaturated hydraulic conductivity
- Steady water flow in unsaturated soil
- Soil-water pressure and total head distributions
- Units of $K$
- Flux and velocity
- Transient water flow

**Unsaturated Soil hydraulic conductivity**

As some of soil pores empty, the ability of soil to conduct $H_2O$ decreases drastically

![Graph showing logK vs h for sand and clay](image1)

![Graph showing logK vs θ for sand and clay](image2)

Thus, $K$ is a function of $h$ or $θ$

Or: $K = f(h)$ and $K = f(θ)$ → highly nonlinear function
As some of the soil pores empty, the ability of soil to conduct water decreases drastically:

Water in unsaturated soil.

Unsaturated hydraulic conductivity decreases as volumetric water content decreases:

- Cross-sectional area of water flow decreases;
- Tortuosity increases;
- Drag forces increase

Thus, the unsaturated hydraulic conductivity is a nonlinear function of \( \theta \) and \( h \).

⚠️ Explain why the saturated hydraulic conductivity for a coarse-textured soil is larger than a fine-textured soil.

**Use capillary tube model (Poiseuille's law) to derive \( K(\theta) \)**
Assume soil pores consist of bundles of capillary tubes of different sizes \( L_c = \) length of twisted capillary, with a length \( L_c > L \) (length of column)

\[
Q_J = \frac{\pi R_J^4 \Delta P}{8Lv} = \frac{\pi R_J^4 \rho_w g \Delta H}{8vL_c} \text{ for a single capillary tube } J
\]

Then for the total volumetric flow rate \( (Q_T) \)

\[
Q_T = \sum_{j=1}^{M} N_J Q_J = \frac{\pi \rho_w g \Delta H}{8vL_c} \sum_{j=1}^{M} N_J R_J^4, \text{ where } N_J \text{ is number of capillaries of radius } R_J
\]

and \( M \) is the number of different capillary size classes in the bundle of capillary tubes.

The total flux \( (J_w) \) can be calculated by dividing through by the total cross-sectional area of the soil column:

\[
J_w = \frac{Q_T}{A} = \frac{\pi \rho_w g \Delta H}{8vL_c} \sum_{j=1}^{M} n_J R_J^4 \text{ where } n_J = \frac{N_J}{A} \text{ is number of capillaries per unit area of radius } R_J
\]
While defining tortuosity as $\tau = \frac{L_c}{L}$, the flux equation can be rewritten as:

$$J_w = \frac{\pi \rho_w g}{8\pi \tau} \sum_{j=1}^{M} n_j R_j^2 \frac{\Delta H}{L} = -K(\theta) \frac{\Delta H}{\Delta X}$$

(with $-\Delta X = L$), and where the unsaturated hydraulic conductivity value can be inferred from the soil water retention curve, with $n_j \pi R_j^2$ denoting the volume of water-filled pores that drain per unit volume of soil at a certain soil-water pressure head ($h_j$).

What determines water flow in soil?

1) Total head gradient
2) $K$ versus $\theta$ relationship

Hence, although the soil is unsaturated, we can still apply the Darcy equation

<table>
<thead>
<tr>
<th>Hydraulic conductivity</th>
<th>Water content</th>
<th>soil-water pressure head</th>
</tr>
</thead>
<tbody>
<tr>
<td>cm/hr</td>
<td>%</td>
<td>cm</td>
</tr>
<tr>
<td><strong>Sand</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>25</td>
<td>-0</td>
</tr>
<tr>
<td>0.15</td>
<td>20</td>
<td>-50</td>
</tr>
<tr>
<td>0.004</td>
<td>10</td>
<td>-100</td>
</tr>
<tr>
<td><strong>Loam</strong></td>
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<td></td>
</tr>
<tr>
<td>2.8</td>
<td>45</td>
<td>0</td>
</tr>
<tr>
<td>1.3</td>
<td>44</td>
<td>-25</td>
</tr>
<tr>
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<td>-14,000</td>
</tr>
<tr>
<td>0.00006</td>
<td>8</td>
<td>-15,000</td>
</tr>
</tbody>
</table>
**What can we learn from these data?**

- Hydraulic conductivity decreases as the water content and/or soil water pressure head decreases;
- In general, for any volumetric water content, the unsaturated hydraulic conductivity for a coarser-textured soil is larger than for a finer-textured soil;
- In general, for any specific soil water pressure head (not close to saturation), the unsaturated hydraulic conductivity for a coarse-textured soil is smaller than for a finer-textured soil.

What other soil hydraulic property is needed to explain this last point?

**Examples of unsaturated steady flow experiments:**

A. Horizontal soil column

![Diagram of a horizontal soil column](image)

- \( b_1 = -20 \text{ cm} \)
- \( X_1 = 0 \text{ cm} \)
- \( z_1 = 0 \text{ cm} \)
- \( h_1 = -20 \text{ cm} \)
- \( H_1 = -20 \text{ cm} \)
- \( X_2 = 30 \text{ cm} \)
- \( z_2 = 0 \text{ cm} \)
- \( h_2 = -30 \text{ cm} \)
- \( H_2 = -30 \text{ cm} \)
- \( L = 30 \text{ cm} \)
- \( b_2 = -30 \text{ cm} \)

Average \( h \) across column is \(-25 \text{ cm}\).

Note that soil is unsaturated (\( h < 0 \)), so we must use an average \( h \) to estimate \( K \). This is acceptable, if changes in \( h \) are small. \( K \) at \(-25 \text{ cm}\) is \(1.3 \text{ cm/hr} \) (see table for loamy soil). Then

\[
J_w = -(1.3 \text{ cm/hr}) \frac{-30 \text{ cm} - (-20 \text{ cm})}{30 \text{ cm} - 0 \text{ cm}} = 0.43 \text{ cm/hr}
\]
**Definition of air-entry value**

For soils A and B, the soil water retention curves and $K(\theta)$-curves are given below. What is the difference between the two soils. Draw also the approximate $K(h)$-curve for soil B.

The air entry value of a specific soil is determined by the radius of the largest pore. If this largest pore is relatively small, the air entry value will be relatively large.

Draw the approximate decrease in h along the horizontal soil column of page 4-5.
Do this for both soils A and B, if the air entry value is -30 cm

Sometimes one uses bubbling pressure to denote air entry value.

- What is the bubbling pressure of a one-bar ceramic porous plate?
- What is the approximate bubbling pressure of the porous cup of a tensiometer?
Remember that simple averaging of $K$ will not work in a column with large $\Delta H$, because $K$ is not constant with $X$, and consequently $\Delta H/\Delta X$ is nonlinear with $X$ (we will get back to this later!).

B. Vertical soil column, with identical soil water pressure head values at boundary conditions:

![Diagram of vertical soil column]

With same average $K$-value as above:

$$J_w = -(1.3 \text{ cm/hr}) \frac{10 \text{ cm} - (-30 \text{ cm})}{30 \text{ cm} - 0 \text{ cm}} = -1.73 \text{ cm/hr}$$

Why is the flux density for the vertical column much larger than for the horizontal column?

Now, change this problem so the water level in the reservoir is at the reference level ($h_2 = -30 \text{ cm}$). Then, water flow is only due to gravity gradient (soil water pressure head gradient is 0), and with $K$ (at -30 cm) approximately equal to 1.25 cm/hr, the flux is

$$J_w = -1.25 \frac{0 - (-30)}{30 - 0} = -1.25 \text{ cm/hr}$$

NOTE: In this case, the soil-water pressure head is constant throughout the column, so the water
content value is uniform as well. This is the best system to set up for determination of unsaturated K in the laboratory, because K is constant with depth.

**Another Example: Two tensiometers in the field (close together):**

Assume that the soil is uniform in the horizontal direction.

**NOTES:** \( H_2 > H_1 \), hence water is flowing from \( H_2 \) to \( H_1 \) (down). Under field conditions the soil surface is chosen for the gravity reference.

Soil-water pressure head is negative, so the soil is unsaturated and \( K \) is a function of \( h \).
\[ h = \frac{-327\text{cm} - 335\text{cm}}{2} = -331\text{cm} \quad \text{“field capacity”} \]

From a hypothetical table we find \( K_{-331} = 0.01 \text{ cm/hr} \); thus \( K(h) \) is known. Then:

\[ J_w = - (0.01 \text{ cm/hr}) \frac{-427 - (-465)}{-100 - (-130)} = -0.0127 \text{ cm/hr DOWN} \]
\[ = -0.3 \text{ cm/day} \]

The downward flux of .3 cm/day is significant, if compared with plant transpiration.

What is the range in plant transpiration (mm/day) in California?

**Soil-water pressure head distributions for steady state flow in unsaturated soils:**

[Diagram of soil-water pressure head distributions]

Problem: How does the soil-water pressure head vary with \( x \)?

NOTE: Both \( h \) and \( K(h) \) changes greatly with \( x \) inside column.

Without calculus: \[ J_w = - K \frac{H_2 - H_1}{x_2 - x_1} \]
With calculus \( J_w = -K \frac{dH}{dx} \)

\[ H = h + z \]
\[ \frac{dH}{dx} = \frac{dh}{dx} + \frac{dz}{dx} \]
\[ J_w = -K \frac{dh}{dx} \]
\[ \int_0^x J_w \, dx = -\int_0^h K \, dh \]

Assume \( K = a + bh \) (cm/day)

Then:

\[
\int_0^x J_w \, dx = -\int_0^h (a + bh) \, dh
\]

\[
J_w \, x \bigg|_0^x = -\left[ ah + \frac{bh^2}{2} \right]_0^h = -ah - \frac{bh^2}{2}
\]

\[ J_w \, x = -ah - \frac{b}{2} h^2 \]

\[ \frac{b}{2} h^2 + ah + J_w \, x = 0 \]

This is a quadratic equation, which can be easily solved for \( h \):

How is the following solution obtained?

\[
h = -\frac{1}{b} \left( a \pm \sqrt{a^2 - 2bJ_w \, x} \right)
\]

The solution has both a positive and negative root. Which one can only be used?

At \( x=0, h = 0 \), so we can only use the negative root:
\[ h = - \frac{1}{b} \left( a - \sqrt{a^2 - 2bJ_w x} \right) \]

This solution yields \( h \) as a function of distance \( x \), or \( h = f(x) \). However, we must first know \( J_w \). This can be determined, if \( K \) as a function of \( h \) is known.

Let \( a = 4, b = 0.02 \). Then use previously found relationship:

\[
\int_0^x J_w \, dx = - \int_0^h (a + bh) \, dh
\]

And substitute the known values of \( h \) at the left and right-hand side of the soil column:

\[
J_w x \bigg|_0^{50} = - \left[ a h + \frac{bh^2}{2} \right]_0^{-180} = 180
\]

Substitute the known values of \( a \) and \( b \), and solve for \( J_w \):

\[
50 \cdot J_w = 180 \left[ 4 + \frac{0.02}{2} (-180) \right] = 7.92 \text{ cm/day}
\]

Now go back and substitute the values of \( a, b, \) and \( J_w \) in above:

\[
h = - \frac{1}{0.02} \left( 4 - \sqrt{16 - 2(0.02)(7.92)x} \right)
\]

The following graph shows the change of \( h \) from the left-hand side (\( x=0 \)) to the right-hand side of the soil column:
Explain the nonlinear change of $h$ with $x$?

A similar derivation can be written for the **vertical case** (with identical boundary conditions):

Assume again that $K = a + bh$ (with $a=4$ and $b=0.02$), and solve for $J_w$ and $h(z)$:

$$J_w = -K \frac{dH}{dX} \quad (X=z)$$

$$H = h + z$$

$$\frac{dH}{dz} = \frac{dh}{dz} + \frac{dz}{dz}$$

$$J_w = -K \frac{dh}{dz} - K$$

$$\frac{J_w}{K} + 1 = -\frac{dh}{dz}$$

$$dz = -\frac{dh}{(1 + J_w / K)} = -\left(\frac{K}{K + J_w}\right)dh$$

Still not in easy form for integration. Change to:

$$dz = -\left(\frac{K + J_w - J_w}{K + J_w}\right)dh = -(1 - \frac{J_w}{K + J_w})dh$$
Substitute for K and integrate:

\[
\int_{0}^{50} dz = - \int_{0}^{180} dh + \frac{J_w}{b} \int_{0}^{180} \frac{bdh}{1 + bh + J_w}
\]

\[
\int \frac{du}{u} = \ln u + c, \text{ let } u = a + bh + J_w
\]

which gives \( du = bdh \)

\[
50 = -180 + \frac{J_w}{b} \ln(a + bh + J_w) \bigg|_{-180}^{0}
\]

\[
230 = \frac{J_w}{b} \ln\left(\frac{a + J_w}{a - 180b + J_w}\right)
\]

\[
230b = \frac{J_w}{b} \ln\left(\frac{a + J_w}{a - 180b + J_w}\right)
\]

\[
a = 4, \quad b = 0.02
\]

\[
\frac{4.6}{J_w} = \ln\left(\frac{4 + J_w}{0.4 + J_w}\right)
\]

Get \( J_w \) by iteration or graphically to obtain: \( J_w = -10.71 \text{ cm/day} \).

Then, to find \( h(z) \):
\[ \int_0^z dz = \int_{-180}^h dh + \frac{J_w}{b} \int_{-180}^h \frac{bdh}{a + bh + J_w} \]
\[ z = -h - 180 + \frac{J_w}{b} \ln \left( \frac{a + bh + J_w}{a - 180b + J_w} \right) \]

which yields with \( a=4 \), \( b=0.02 \) and \( J_w=-10.71 \):
\[ z = -180 - h - 535.5 \ln(0.651 - 0.00194h) \]

We can't solve for \( h \) directly, but \( h(z) \) can be obtained by substitution of \( h \)-values between 0 and -180, to compute the appropriate \( z \)-values.

**Head Profiles**

- Describe the distribution of gravitational head (\( z \)), soil-water pressure head (\( h \)), and total head (\( H \));

- Used to determine direction of water flow in a soil system

\[ H = h + z \] , where \( z \) is determined by the height of the point of interest relative to some reference plane,

**Gravitational head**

If the **reference level is taken at the bottom of a soil column** (or below the soil surface), we plot the gravitational head as follows:

![Gravitational head graph](image-url)
If the reference level is taken at the top of the sample (or at the soil surface), the slope of the line is the same, but the values of the gravitational head are negative as follows:

**Below the water table; No water Flow: \( H_1(0 \text{ cm}) = H_3(50 \text{ cm}) \)**

*Note:* in this example in the field or in a column, the reference level is selected at 50 cm below the water table. You can place a gravity reference anywhere as long as it was not already specified in a problem and provided you keep the same location throughout the working of the problem.
No water flow (hydraulic equilibrium) above water table; reference level at water table.

![Diagram showing soil-water pressure head, total head, and gravitational head.]

What is the soil water pressure head at the water table?

![Graph showing volumetric water content (θ) vs. soil depth.]

Which soil hydraulic property is needed to determine the change of volumetric water content with soil depth from the hydraulic equilibrium profile?
Steady rainfall, steady-state vertical flow ($J_w$ constant)

Given that:

- Water flow is steady state in the downward direction;
- Water table at 0 cm ($h=0$);
- Reference level is selected: elevation is 0 cm;
- The soil-water potential at the soil surface is measured and known (see symbol);

Plot the distribution of $h$ and $H$ with position.

The "no flow" line is the hypothetical $h$-line if indeed water flow is not occurring. The "Constant K" line is the hypothetical $h$-line if $K$ is indeed a constant throughout the column. However, $K$ cannot be a constant for an unsaturated column except for the case where $h$ is maintained everywhere the same within the column or $h >$ air entry value. The vertical broken line at $h = -25$ cm further defines the region in which the real $h$ must lie. The real $h$ must fall within the area bounded by the "Constant K", the "no flow", and the vertical broken line.

At top of soil: $h$ is the smallest (most negative), hence the $K$ is the smallest. Therefore, the hydraulic head gradient ($\Delta H/\Delta X$) is largest there.

At the bottom of the soil: $h$ is the largest (least negative), hence the $K$ is the largest. Therefore, the hydraulic head gradient ($\Delta H/\Delta X$) is smallest there.

Steady state evaporation ($J_w$ constant)
As for the steady state rainfall situation:

At top of soil: \( h \) is the smallest (most negative), hence the \( K \) is the smallest. Therefore, the hydraulic head gradient (\( |\Delta H/\Delta X| \)) is largest there.

At the bottom of the soil: \( h \) is the largest (least negative), hence the \( K \) is the largest. Therefore, the hydraulic head gradient (\( |\Delta H/\Delta X| \)) is smallest there.

**Determining shape of \( h \) and \( H \) versus. \( X \) for steady flux situations**

1. Plot \( z \), \( h \) and \( H \) at top and bottom of profile (\( H=h+z \))
2. If \( h \geq 0 \) (i.e. saturated), draw straight lines. (Lines are also straight inside the capillary fringe).
3. For \( h < 0 \), determine whether \( K \) is smaller or larger at one end as compared to the other. \( K \) is largest where \( h \) is greatest.
4. Since \( J_w \) must be everywhere constant, determine whether \( |\Delta H/\Delta X| \) is larger or smaller at one end as compared to the other end. If \( K \) small, then \( |\Delta H/\Delta X| \) must be large. If \( K \) large, then \( |\Delta H/\Delta X| \) must be small.
5. By knowing the relative magnitudes of \( \Delta H/\Delta X \) throughout the profile, plot \( H \) versus \( X \) by changing slope accordingly. (The curve drawn is only an approximation.)
6. With \( H \) versus \( X \), you can get \( h \) versus. \( X \). **TOTAL HEAD IS DRAWN FIRST!**

\( \heartsuit \) The soil for which the diagram below is given has a saturated hydraulic conductivity of 10 cm/day, and has a constant water table at 1 m below the soil surface. Explain what you
see in the diagram, which presents soil-water pressure head distributions for a range of flux density values.

And has the following soil water retention curve:

What are units of K?
Using potential per unit mass

\[ J_w = -K \frac{\Delta \mu T}{\Delta X} \]

\[ \frac{m}{\text{sec}} = - ? \frac{J}{\text{kg m}} \]

\[ K = \frac{m \text{sec}^{-1}}{J \text{kg}^{-1} \text{m}^{-1}} = \frac{\text{kg} m^2}{J \text{sec}} \]

\[ = \frac{\text{kg} m^2}{Nm \text{sec}} = \frac{\text{kg} m^2 \text{sec}^2}{\text{kg} m^2 \text{sec}} \]

\[ \therefore K = \text{sec} \]

Using potential per unit weight (head)

\[ J_w = -K \frac{? H}{? X} \text{ or } \frac{\text{cm}=? \text{cm}}{\text{sec} \text{ cm}} \]

thus, units of \( K = \text{cm/sec} \)

What are the units of \( K \), if using potential per unit volume?
Flux and velocity

\[ J_w = \frac{V}{At} = \frac{cm^3 H_2O}{cm^2 soil \ surface \ sec} \]

What happens to velocity of \( H_2O \) when flowing from a large to a smaller diameter tube?

\[
\text{It increases by a factor of } \left[ \frac{r_2}{r_1} \right]^2 \text{ where } r_1 < r_2
\]

That is,

\[ Q = v_2A_2 = v_1A_1, \text{ and } A = \pi r^2 \text{ for a cylindrical tube. Then } v_1 = v_2\frac{A_2}{A_1} = v_2\left(\frac{r_2}{r_1}\right)^2 \]

The same occurs in soils when water flows from large to small pores:

Cross-sectional area through which water moves decreases as it moves into the soil. Therefore, the water in the soil moves faster than on top of soil.

\[ v = \frac{J_w}{\theta} = \text{average pore water velocity} \]
**Intrinsic permeability, k**

Removes fluid properties, such as density and viscosity, as factors influencing water flow, and yields a constant (property of the soil only).

This soil characteristic is defined as the soil permeability: \( k = \frac{K}{f} = \frac{\text{cm} / \text{sec}}{1 / \text{cm sec}} = \text{cm}^2 \)

**Fluidity is fluid property**, \( f \) - ability of liquid or gas to flow, based on its viscosity and density;

\[
f = \frac{\rho g}{\nu}, \quad \text{where } \nu \text{ is dynamic viscosity, } \frac{N \text{ sec}}{m^2}\]

**Transient Vertical Water Flow**

\[
J_w = -K \frac{dH}{dz} = -K \frac{d(h + z)}{dz}
\]

\[
J_w = -K \frac{dh}{dz} - K \text{ at steady - state}
\]

Transient state must include time - Equation of Continuity

\[
\frac{\partial \theta}{\partial t} = - \frac{\partial J_w}{\partial z} \quad \text{IN}
\]

\[
\Delta \theta \quad \text{IN}
\]

\[
J_w \text{ OUT}
\]
\[
\frac{\partial \theta}{\partial t} = - \frac{\partial}{\partial z} \left( -K \frac{\partial h}{\partial z} - K \right)
\]

\[
\frac{\partial \theta}{\partial t} = - \frac{\partial}{\partial z} \left( -K \frac{\partial h}{\partial z} \right) + \frac{\partial K}{\partial z}
\]

Convert \( h \) to \( \theta \)

\[
\frac{\partial h}{\partial z} = \frac{\partial h}{\partial \theta} \frac{\partial \theta}{\partial z}
\]

\[
\frac{\partial \theta}{\partial t} = - \frac{\partial}{\partial z} \left( -K \frac{\partial h}{\partial \theta} \frac{\partial \theta}{\partial z} \right) + \frac{\partial K}{\partial z}
\]

Let \( D(\theta) = K \frac{\partial h}{\partial \theta} \)

\( \frac{\partial h}{\partial \theta} \) is slope of soil-water characteristic curve, and \( D \) is soil water diffusivity (m\(^2\)/sec)

\[
\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left( D \frac{\partial \theta}{\partial z} \right) + \frac{\partial K}{\partial z}
\]

Richards Equation

**Solutions**

- Partial differential equations
- Boundary value problems
- Numerical Techniques
  - Finite difference
  - Finite element