Chapter 3. Saturated Water Flow

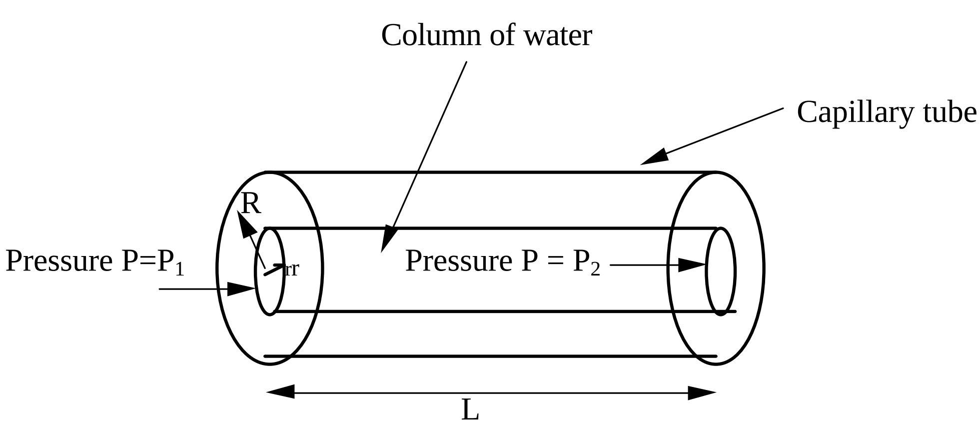
- All pores are filled with water, i.e., volumetric water content is equal to porosity ($\theta = \theta_s$ with $\theta_s = \phi$)
- Nonequilibrium. Water flows from points of high to points of lower total water potential
- Total water potential is sum of gravitational and soil water pressure potential, or

  $\psi_T = \psi_p + \psi_z$ (J/m^3) or $H = h + z$ (m)

- Consider steady-state water flow. I.e., water flow does not cause changes in water storage values (constant flow rate and volumetric water content at any position (X) does not change with time).

  This is opposed to transient water flow where H and θ change as a function of time.

<u>Water flow in capillary tube:</u>



Section of capillary tube of radius R and length L, filled with water flowing in response to a pressure difference P_1 - P_2. Force balance is done on water cylinder of radius r.

Consider volume element of radius r, with r = 0 at center of tube (Figure 3.1 on page 74 book).

Pressure difference across tube = $P_1 - P_2$ (N/m^2).

Do force balance on cylindrical water volume with r radius r < R.

Pressure force across two ends = $(P_1 - P_2).A = \Delta P \pi r^2$

Shear force on external area of water volume = $\tau(2\pi rL)$

Equating these 2 forces leads to: $\tau = \Delta Pr/2L$
Earlier we found that $\tau = -\upsilon\, dv/dr$

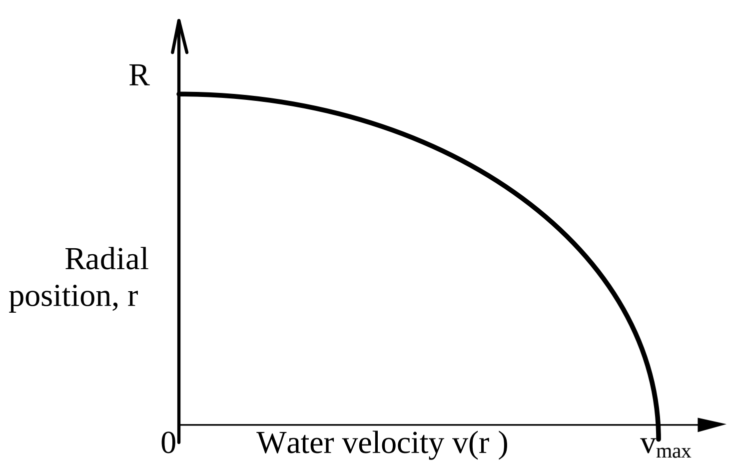
Equate and put terms with r together on left-hand side and integrate from r to R (v=0):

$$rdr = -\frac{2Lu}{\Delta P}dv$$

$$\int_r^R rdr = \frac{-2Lu}{\Delta P}\int_v^0 dv$$

$$yields \quad v(r) = \frac{\Delta P}{4Lu}(R^2 - r^2)$$

When plotting velocity as a function of radius r, its distribution is parabolic:



✪ For which value of r is the water velocity at its maximum ?

$$\text{At } r = 0, \text{ where } v = v_{max} = R^2\Delta P/4L\nu$$

To find volumetric flow rate Q (volume per unit time), we must integrate v(r) over the area of the cylinder. Use cylindrical coordinates:

$$\boldsymbol{Q = \int\int v(r)dA}$$

$$\textit{\textbf{Yielding}}\;\; \boldsymbol{Q} = \frac{p\,\boldsymbol{R^4\,\Delta P}}{\boldsymbol{8L}u}$$

Using flow rate per unit area (water flux)

$$\boldsymbol{J_w} = \frac{\boldsymbol{R^2\,\Delta P}}{\boldsymbol{8L}u}\;\left[= \frac{\boldsymbol{R}^2 r_w\, \boldsymbol{g\Delta H}}{8\boldsymbol{L}n} = \boldsymbol{K}\frac{\boldsymbol{\Delta H}}{\boldsymbol{L}}\right]$$

NOTE: This equation is known as Poiseuille's law (Water flux is a function of pore radius)

✪ Would you expect differences in flow rates between sands and clays, and if so why ?

In 1856, Darcy found for flow through saturated sand that

$$\boldsymbol{Q = \frac{V}{t}}\;\; \textit{\textbf{and is proportional to}}\;\; \boldsymbol{\frac{\Delta H}{\Delta X}}$$

$$\boldsymbol{Q}\; - \;\textit{\textbf{discharge rate (volume water per time)}}$$

$$\boldsymbol{V}\; - \;\textit{\textbf{volume of}}\;\boldsymbol{H_2O}$$

$$\boldsymbol{t}\; - \;\textit{\textbf{time}}$$

$$\boldsymbol{X}\; - \;\textit{\textbf{distance or position}}$$

$$\boldsymbol{\frac{\Delta H}{\Delta X}}\; - \;\textit{\textbf{Total head gradient}}$$

A gradient is the change of some parameter with position.

The gradient should really be taken as an infinitesimal small change, $\frac{dH}{dX}$ (differential notation)

Volume of H_2O flowing through soil will be dependent upon area, hence

Divide Q by A to describe flow on a relative basis:

$$\frac{Q}{A} = J_w \left[\frac{m^3}{m^2\ s}\ or\ \frac{m}{s}\right] \qquad \frac{Q}{A} = J_w = \frac{V}{At}$$

$$J_w \text{ - flux density (flux), which is proportional to } \frac{\Delta H}{\Delta X}$$

The proportionality constant between flux and gradient is K, the hydraulic conductivity, or

$$J_w \text{ is proportional to } K\frac{\Delta H}{\Delta X}$$

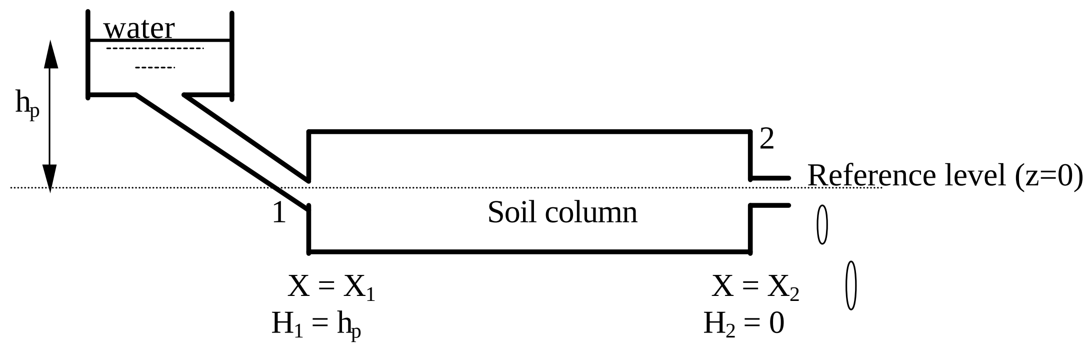
For saturated sand, Darcy found that K was constant. Thus,

$$J_w = K\frac{\Delta H}{\Delta X}$$

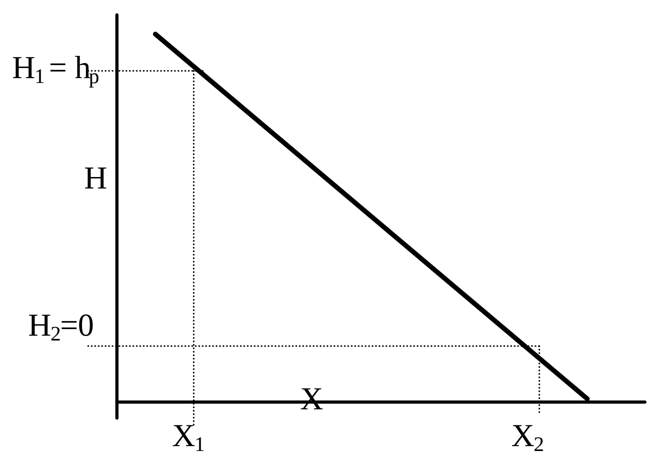
✪ Can you now relate the Darcy flow equation with Poiseuille's law ?

What sign? + or -

Consider a horizontal soil column, saturated with water:



And plot the change of total head (H) with position (X):



Then:

$$J_w = K \frac{H_2 - H_1}{X_2 - X_1} \qquad \begin{matrix} H_2 < H_1 \\ X_2 > X_1 \end{matrix}$$ (Note: Only in this example)

Thus, $\dfrac{\Delta H}{\Delta X}$ is negative.

For above case, J_w would be negative, whereas flow is from left to right

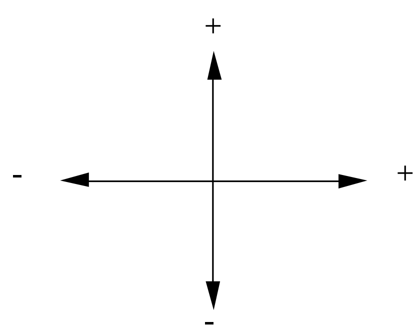
- Since flow is towards the right or towards larger X, it would be logical to have J_w be positive.

- Put minus sign in front, so that:

$$J_w = - K \frac{\Delta H}{\Delta X}$$

So that flux is positive, if water is moving towards increasing X, or H is decreasing as X is increasing .

Directional Convention for position (X)and flux density (J_w)



Convention for locating variables is dependent on column orientation

	horizontal	vertical
X_1	left (x_1)	bottom (z_1)
X_2	right (x_2)	top (z_2)
H_1	left	bottom
H_2	right	top

NOTE: In this class when using Darcy's Law and other similar transport equations, use this convention

✪ When using this convention correctly, what is the sign of J_w if water flow is downward, upward, left or right ?

J_w = + flow to right or up

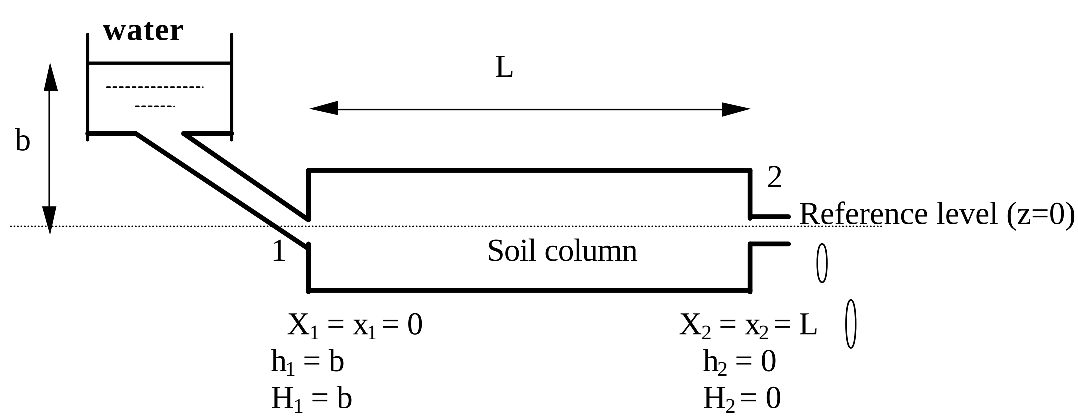
J_w = - flow to left or down

The proper sign of J_w is your self-check of the problem. Remember if $H_2 > H_1$, water flows from position 2 towards position 1.

The value of GRAVITATIONAL HEAD (z) depends on where you select the reference level. Above the reference level it is positive, below it is negative.

EXAMPLES:

A. Steady state saturated flow in horizontal soil columns

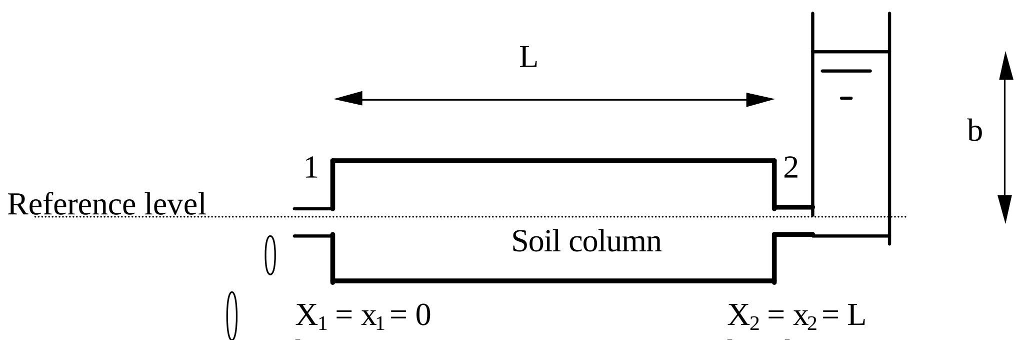


Steady state is achieved when the flux of water out of the column is constant over time. If we could measure the flux at several cross sections at each point in the column, its value must be constant (flux of water in column = flux of water out column; hence there is no water storage change at any location in the column).

$$J_w = -K\frac{dH}{dX} = -K\frac{H_2 - H_1}{X_2 - X_1} = -K\frac{0 - b}{L} = \frac{Kb}{L}$$

And thus water flow is from left to right !!!

✪ In which direction would water flow if the total head gradient is reversed ???



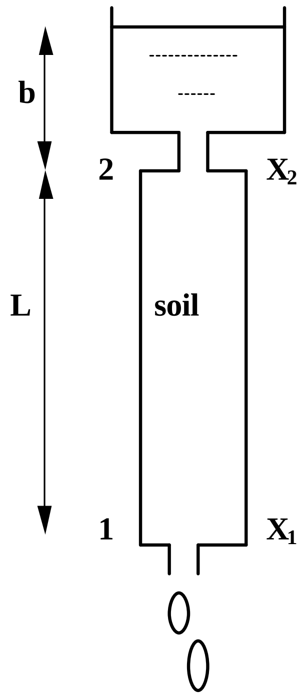
$X_1 = x_1 = 0$
$h_1 = 0$
$H_1 = 0$

$X_2 = x_2 = L$
$h_2 = b$
$H_2 = b$

$$J_w = -K\frac{dH}{dX} = -K\frac{H_2 - H_1}{X_2 - X_1} = -K\frac{b - 0}{L} = -\frac{Kb}{L}$$

If the head gradient is reversed; water flow is from right to left !!!!!!

B. Vertical soil columns:



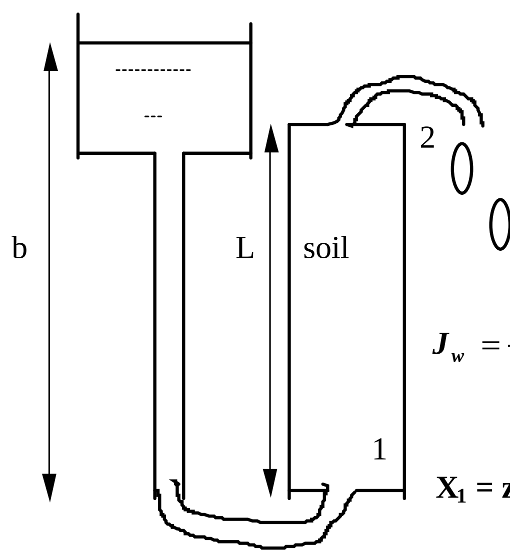
$X_2 = z_2$=L ; h_2 = b ; H_2 = L+b

$$J_w = -K\frac{dH}{dX} = -K\frac{H_2 - H_1}{X_2 - X_1} = -K\frac{b+L}{L}$$

downward flow

$X_1 = z_1$=0 ; h_1 = 0 ; H_1 = 0

When head gradient is reversed:



$X_2 = z_2$=L ; h_2 = 0 ; H_2 = L

$$J_w = -K\frac{dH}{dX} = -K\frac{H_2 - H_1}{X_2 - X_1} = -K\frac{L-b}{L} = K\frac{b-L}{L}$$

upward flow (J_w >0)

$X_1 = z_1$=0 (reference level) ; h_1 = b ; H_1 = b

NOTE: X-directional convention is independent of gravity reference

✪ Compute the flux density, if one assumes the reference level at top of the column.

<u>How does the total head change with position, if the soil is uniform?</u>

Consider the DARCY equation and note that it assumes that flow is steady state:

$$J_w = -K\frac{dH}{dX}$$

- **Time does not appear in Darcy equation**
- **Hence, flux (J_w) is constant**
- **And** q **and h (or H) do not change with time**
- **K is constant, because soil is uniform**

Then, for a uniform soil dH/dX is constant, and it can be shown that therefore H varies linearly with position X (straight line).

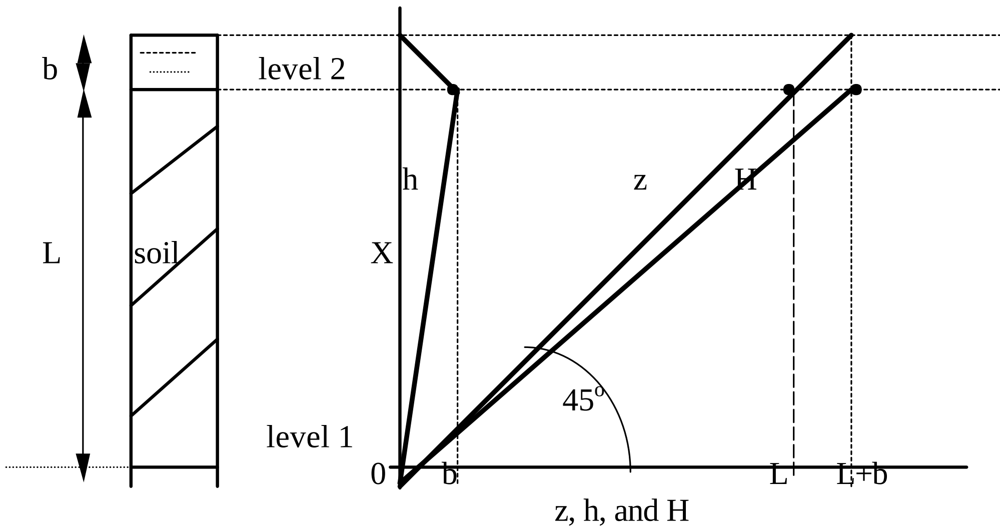
✪ Proof that H varies linearly with X for a uniform soil.

dH/dX is constant, or dH/dX = C_1

Integrate: $\int dH = C_1 \int dX$

Or: $H = C_1X + C_2$, for which the values of C_1 and C_2 depend on the values of H at the column ends.

Saturated Conductivity Measurement:
(Constant Head Method)



Draw z-line, which is a 1:1-line (slope is 45°).

Draw H-line by adding z to h at any position X where both h and z are known.

$h_1 = 0$ $z_1 = 0$ $H_1 = 0$ $X_1 = 0$

$h_2 = b$ $z_2 = L$ $H_2 = b+L$ $X_2 = L$

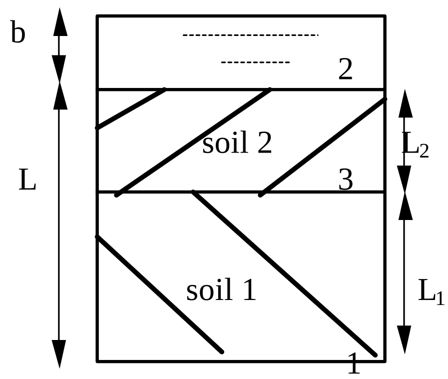
Also h varies linearly with X.

Applying Darcy's flow equation, and solve for K_s (saturated hydraulic conductivity):

$$J_w = \frac{Q}{A} = -K_s \frac{H_2 - H_1}{X_2 - X_1} = -K_s \frac{b+L}{L}$$

$$K_s = -\frac{QL}{A(L+b)}$$

Saturated flow in layered soils:



$h_2 = b$	$z_2 = L$	$X_2 = L$	$H_2 = b+L$
$h_3 = ?$	$z_3 = L_1$	$X_3 = L_1$	$H_3 = ?+L_1$
$h_1 = 0$	$z_1 = 0$	$X_1 = 0$	$H_1 = 0$

✪ If K_s-values for soil layers 1 and 2 are known, compute the soil water pressure head at the interface of the two layers (position 3).

Steady-state:

$$J_w \;=\; = \;-K_1 \frac{H_3 - H_1}{X_3 - X_1} \;=\; -K_2 \frac{H_2 - H_3}{X_2 - X_3} \;=\; -K_{eff} \frac{H_2 - H_1}{X_2 - X_1}$$

- Solve for H_3, and subsequently for h_3
- Check that J_w (layer 1) = J_w (layer 2)

From electical analog: R = $\Delta V/I$ (electrical resistance), we can write for hydraulic resistance (R_H) for each soil layer i:

For each layer i:

$$R_{H,i} = \frac{potential\ difference}{flux} = \frac{DH_i}{J_w} = L_i/K_i$$, from Darcy equation

where ΔH_i denote the total head difference across layer i.

Remember from electrical theory that the total resistance is equal to the sum of the individual resistances when in series. Hence, for the total layered soil system (with n layers), we can then define the effective hydraulic resistance ($R_{H,eff}$)

$$\boldsymbol{R_{H,eff} = \sum R_{H,i} = \sum_{i=1}^{n} \frac{L_i}{K_i}}$$

Also, we define: $\boldsymbol{R_{H,eff} = \frac{L}{K_{eff}}}$

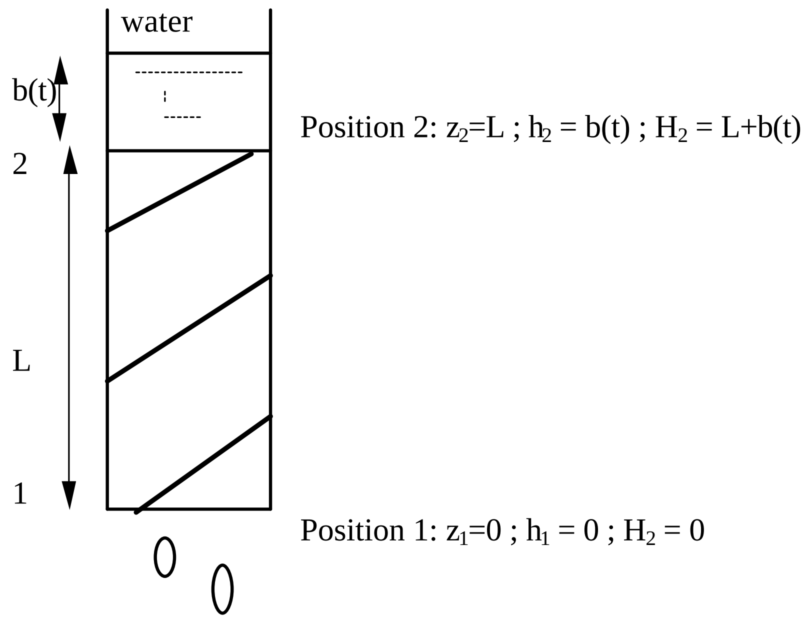
So that we can compute the effective hydraulic conductivity for the layered soil system (K_{eff}) by equating the two above expressions (See book, page 84):

$$\boldsymbol{K_{eff} = \frac{L}{\sum \frac{L_i}{K_i}}}$$

✪ Explain how we can apply the effective hydraulic conductivity concept to compute the steady state flux by knowing the total head values at the top and the bottom of a multi-layered soil column, and the thickness and saturated hydraulic conductivity values of each individual layer.

Falling head method to measure the saturated hydraulic conductivity of uniform soil:

This method is used if it is expected that K_s is low



- The soil is saturated and has an unknown saturated hydraulic conductivity value, K_s
- The head of water on top of the soil column decreases with time (t), and is equal to b(t)
- For uniform size column, the flux coming out of the column is:

$$J_w = \frac{db}{dt} = -K_s \frac{b(t)+L}{L}$$

Or: $$\frac{db(t)}{b(t)+L} = -\frac{K_s dt}{L},$$ where $b=b_0$ at $t=0$ and $b=b_1$ at $t=t_1$

- Integrate and solve for K_s at time t_1: $$K_s = \frac{L}{t_1}\ln\left[\frac{b_o+L}{b_1+L}\right]$$

✪ Special assignment: Derive the expression for K_s, if the area of the water-filled tube (a) is much smaller than the area of the soil column (A)